Atmospheric OPD implications for adaptive IR and optical interferometry

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ABSTRACT

We present the concept of piston angular anisoplanatism and derive the temporal evolution of the optical path differences between the two pupils of an astronomical interferometer due to random fluctuations of the atmospheric refraction index, obtaining expressions for the temporal power spectra of phase fluctuations caused by differential piston and photon noise. This allows us to evaluate the residual phase variance left in a fringe tracking servo-loop system and obtain estimates for the sky-coverage due exclusively to differential piston effects, concluding that in the V-band the percentage of sky that can be observed is a few percent while in the K-band we should be able to cover the entire sky.

1. INTRODUCTION

For a long time it has been recognized that optical path differences originated from random fluctuations of the atmospheric index of refraction may cause a fringe displacement which if not corrected can completely blur and cancel out the visibility function. Thus, current small-aperture interferometers in operation already incorporate active fringe tracking systems which use the science object to track the fringes. These interferometers are very limited in sky-coverage due mainly to their small apertures which impose rather strong constraints on the limiting magnitude ($m_v = 5.7$). On the other side, their small apertures of the order or $r_0$ makes non compulsory the use of adaptive optics. This situation is completely different for the new generation of optical ground-based interferometers1–3 with apertures of 8-m and 10-m diameter and adaptive optics systems to correct from atmospheric degradation before cophasing the wavefronts from each single aperture. However, whereas tip-tilt and higher-order atmospheric effects have been studied in detail (e.g see Ref. 4–6) the effects of atmospheric differential piston on adaptive optics interferometry have been somehow overlooked and rarely found in the literature.7, 8

2. EFFECT OF THE DIFFERENTIAL PISTON ON THE VISIBILITY

Let us assume an ideal situation where random fluctuations of the atmospheric refractive index distort the incoming wavefront by introducing a constant offset but of different values on the portions of wavefronts intercepted by each of the pupils of an astronomical interferometer*. Assuming identical atmospheric transmissions up to the two interferometers and along its two arms, the fringe pattern on the image plane is given by:

$$ I(\vec{r}) = 2I_{DL}(\vec{r})(1 + V \cos(\phi(\tau) - \phi_{DP})) $$

(1)

where $\vec{r}$ is a vector on the image plane, $I_{DL}$ is the diffraction limited pattern due to a single aperture (i.e. an Airy disk if we assume circular apertures), $V$ is the instant visibility, $\phi_{DP}$ the phase caused by the differential piston (and therefore the DP subscript) and $\phi(\tau)$ the phase due to both spatial and temporal coherence. Once we have selected the point of observation in the image plane $\phi(\tau)$ has a constant value so that the averaged visibility $\bar{V}$ will differ from the instant visibility due to temporal changes of $\phi_{DP}$. Assuming that $\phi_{DP}$ behaves as a Gaussian random variable with variance $\sigma_\phi^2$ it can be readily found that the time average visibility function $\bar{V}$ is given by:

$$ \bar{V} = V \exp(-\frac{\sigma_\phi^2}{2}) $$

(2)

*The difference of these offsets is referred to as differential piston in what follows.
3. DIFFERENTIAL PISTON ANGULAR ANISOPLANATISM

The loss of correlation between the differential piston when looking towards the on-axis science object and the differential piston measured from an off-axis bright reference source translates into a residual phase difference referred to as differential piston angular anisoplanatism. Adopting the same terminology as in Ref. 9 we consider an interferometer made up of two pupils of diameter $D$ and a center-to-center distance (i.e. baseline) equal to $\Delta$. The instrument is assumed to be pointing towards zenith and observing an on-axis scientific target while correcting for differential piston by observing a bright reference source at an off-axis angle $\theta$. Further, consider an atmosphere with a Von Karman power spectrum with outer scale $L_0$, all turbulent power concentrated in a single layer at an altitude $h$ above the interferometer and a Fried parameter $r_0$ at the observing wavelength $\lambda$. Referring to Fig. 1-left for the definition of the geometrical elements $s$, $d_{12}$ and $d_{21}$, it can be shown that the phase variance due to differential piston angular anisoplanatism ($\sigma_\phi^2_{\text{ani}}$) is given by:

$$
(\sigma_\phi^2_{\text{ani}}) = 0.3664 \pi^{8/3} \left( \frac{D}{r_0} \right)^{5/3} \int_0^\infty x \left[ x^2 + \left( \frac{D}{r_0} \right)^2 \right]^{11/6} J_1^2(x) \ dx
$$

(3)

In Fig. 1-right we show the behavior of the anisopistonic optical path difference rms $(\sigma_p)_{\text{ani}} = 2\pi (\sigma_\phi)_{\text{ani}}/\lambda$ for an LBT-like configuration ($D = 8.4$ m, $\Delta = 14.4$ m) due to a single turbulent layer with $r_0(0.55\mu\text{m}) = 0.2$ m at $h = 10$ km above the telescope for several $L_0$ values.

![Figure 1](image)

**Figure 1.** Left: geometrical elements considered for the calculations of phase variance due to differential piston angular anisoplanatism. Right: anisopistonic optical path difference rms $(\sigma_p)_{\text{ani}}$ as a function of the off-axis angle of the reference source for a single-layer turbulent atmosphere (see text for the details on the rest of parameters). Plotted curves refer to $L_0 = 10, 20, 50, 100$ m and $\infty$ from the least to the steepest curve. Solid and dotted lines represent the cases where the reference source is displaced along and orthogonally to the baseline, respectively.

4. SERVO-LOOP DELAY EFFECTS

The second source of differential piston phase residuals in an adaptive optics interferometer is due to the fact that the correction is applied with a certain delay during which little changes are expected to occur. In order to estimate the contribution from this source to the total differential piston variance we must first compute the temporal power spectrum of the differential piston. Once such temporal power spectrum is obtained, the temporal behavior of phase residuals in our servo-loop is obtained by considering the servo-loop transfer function.

4.1. Temporal Power Spectrum of Differential Piston Phase Fluctuations

To do so we have first obtained the temporal auto-covariance function of the differential piston under rather general assumptions, namely: Taylor hypothesis and spatial isotropy of the field of random atmospheric refractive index.
fluctuations\(^1\). After obtaining the temporal auto-covariance we are ready to find the temporal power spectrum by just applying the Wiener-Khinchin theorem. Assuming a Von Karman spectrum to model the random fluctuations of the atmospheric index of refraction, the expressions for the temporal power spectra corresponding to the cases of wind parallel and perpendicular to our interferometer baseline are:

\[
\Gamma_{\phi}^{\parallel}(f) = \frac{0.6202}{\lambda^2 D^2} \sec \psi f^{-\frac{14}{3}} \int_0^\infty dh C_n^2(h) v(h) \frac{f^2}{v^2(h)} F^{\parallel} \left( \frac{f}{v(h)} ; D, L_0 \right) \left[ 1 - \cos \left( \frac{2\pi \Delta}{v(h)} \right) \right] \tag{4}
\]

\[
\Gamma_{\phi}^\perp(f) = \frac{0.6202}{\lambda^2 D^2} \sec \psi f^{-\frac{14}{3}} \int_0^\infty dh C_n^2(h) v(h) \frac{f^2}{v^2(h)} F^{\perp} \left( \frac{f}{v(h)} ; D, \Delta, L_0 \right) \tag{5}
\]

where \(\psi\) is the zenith angle, \(D\) the aperture diameter, \(\Delta\) the interferometer baseline, \(C_n^2(h)\) and \(v(h)\) the structure constant of the refractive index fluctuations and wind velocity at height \(h\), respectively, and \(F^{\parallel}\) and \(F^{\perp}\) are defined according to the following integrals:

\[
F^{\parallel}(y; D, \Delta, L_0) = \int_0^1 dx \frac{x \frac{f}{y L_0}}{\sqrt{1-x^2}} \left[ 1 + \left( \frac{x}{y L_0} \right)^2 \right]^\frac{-\frac{14}{3}}{2} J_1^2 \left( \frac{\pi D}{x y} \right) \tag{6}
\]

\[
F^{\perp}(y; D, \Delta, L_0) = \int_0^1 dx \frac{x \frac{f}{y L_0}}{\sqrt{1-x^2}} \left[ 1 + \left( \frac{x}{y L_0} \right)^2 \right]^\frac{-\frac{14}{3}}{2} J_1^2 \left( \frac{\pi D}{x y} \right) \left[ 1 - \cos \left( \frac{2\pi \Delta}{x y} \sqrt{1-x^2} \right) \right] \tag{7}
\]

In Conan et al.\(^8\) the authors provide the expression of the temporal power spectrum of differential piston phase fluctuations but in the more simple case of a Kolmogorov spectrum and wind parallel to baseline. Our expression for \(\Gamma_{\phi}^{\parallel}\) reduces to theirs for the parallel case with a single layer and \(L_0 \to \infty\). Additionally we have tested our results by comparing against simulations conducted with LA\(^3\)OS\(^2\) code.\(^10\) We conclude this section by remarking the fact that considering the particular parallel and perpendicular cases removes no generality as far as we can always express the differential piston phase power spectrum for any wind direction in terms of \(\Gamma_{\phi}^{\parallel}(f)\) and \(\Gamma_{\phi}^{\perp}(f)\). To do this in addition of a vertical wind speed profile we must also consider the vertical profile of wind directions \(\theta(h)\) with respect to the interferometer baseline.

4.2. Convolving with the Servo-loop Transfer Function

We assume the servo-loop of an adaptive fringe tracking system as made of a correcting element (i.e. delay line), a fringe sensor, an integrator and a feedback element to close the loop. In our calculations we assume a correcting element fast enough not to introduce any significant delay in the correction so that we can approximate \(H_e(f) = 1\). The transfer function for the fringe sensor \(H_s(f)\) takes into account the integration time \(\tau\) plus the delay associated to it, so it is described by (e.g. see Ref. 6) \(H_s(f) = (1 - \exp(-j2\pi f\tau))/(j2\pi f\tau)\). We consider a pure integrator with transfer function \(H_i(f) = 1/(j2\pi f\tau)\) and a unit feedback element (i.e. \(H_f(f) = 1\)). The residual differential piston phase rms \(\sigma_{\phi}\) in the servo-loop system is given by:

\[
(\sigma_{\phi}^2)_{\tau} = 2 \int_0^\infty df |T_{\phi}(f)|^2 \Gamma_{\phi}(f) \tag{8}
\]

where the subscript \(\tau\) makes reference to the integration time and \(|T_{\phi}(f)|^2\) is the square modulus of the servo-loop error-input transfer function given by (e.g. see Ref. 6):

\[
|T_{\phi}(f)|^2 = \left| \frac{1}{1 + H_e(f) H_i(f)} \right|^2 = \frac{(2\pi f\tau)^4}{(2\pi f\tau)^4 - 4\sin^2(\pi f\tau)[(2\pi f\tau)^2 - 1]} \simeq \frac{(2\pi f\tau)^2 + \frac{4}{5}(2\pi f\tau)^4}{1 + \frac{4}{5}(2\pi f\tau)^4} \tag{9}
\]

\(^1\)The Taylor hypothesis, or frozen turbulence approximation, assumes that each turbulent layer moves rigidly without changing its shape. By adopting this hypothesis we can directly link time variations of any quantity related to the wavefront with its spatial variations.
Performing the integration involved in Eq. 8 with \( |T_p(f)|^2 \) and \( \Gamma_\phi(f) \) given by Eq. 9, Eq. 4 and Eq. 5 is an extremely cumbersome task. Therefore we proceed by using approximate expressions for the error-input transfer function in close-loop operation and instead of Eq. 4 and Eq. 5 we make use of the asymptotic behaviours of \( \Gamma_\phi^\parallel(f) \) and \( \Gamma_\phi^\perp(f) \) in Ref. 11. Now the integration in Eq. 8 can be readily done to obtain the residual differential piston phase variance according to the expression:

\[
(s^2_\phi)_{\tau} = A_0 \sec \psi \left( \frac{2\pi}{\lambda} \right)^2 D^{-1/3} \tau^2 \left\{ (1 + A_1) v_2 + \int_0^\infty dh C_n^2(h) v^2(h) \left[ A_2 \cos^2 \theta(h) + A_2 \sin^2 \theta(h) \right] \right\}
\]

(10)

where let us recall that \( \tau \) is the integration time, \( v_2 = \int_0^\infty dh C_n^2(h) v^2(h) \) is the 2nd order velocity moment and where for a Kolmogorov spectrum (i.e. \( L_0 = \infty \)): \( A_0 = 5.81, A_1 = 0, A_2 = 0.69(D/\Delta)^{1/3} \) and \( A_3 = -0.16(D/\Delta)^{1/3} \).

For finite outer scales \( A_0 = 0.643 \) and:

\[
A_1 = \begin{cases} 
0 & \text{if } L_0 < \frac{3}{2} D \\
3.61 \times 10^{-2} \left( \frac{L_0}{D} \right)^{8/3} F_1(4/3, 3/2; 5/2; -0.09(L_0/D)^2) & \text{if } L_0 > \frac{3}{2} D
\end{cases}
\]

(11)

\[
A_2 = \begin{cases} 
3.61 \times 10^{-2} \left( \frac{L_0}{D} \right)^{8/3} \left[ F_1(4/3, 3/2; 5/2; -0.09(L_0/D)^2) + 9.02 \times 10^{-2}(L_0/\Delta)^2 \right] & \text{if } L_0 < \frac{3}{2} D \\
-1.07 \times 10^{-2} \left( \frac{L_0}{\Delta} \right)^{8/3} \left( 1 - 0.474(D/\Delta)^{1/3} \right) F_1(4/3, 3/2; 5/2; -0.04(L_0/\Delta)^2) & \text{if } L_0 > \frac{3}{2} D
\end{cases}
\]

(12)

\[
A_3 = \begin{cases} 
2.31 \times 10^{-2} \left( \frac{L_0}{D} \right)^{11/3} & \text{if } L_0 < \frac{3}{2} D \\
-8.31 \times 10^{-2} \left( \frac{L_0}{D} \right)^{8/3} \gamma(3/2, 3\Delta^2/(4D^2)) \exp(-8.32\Delta^2/L_0^2) & \text{if } L_0 > \frac{3}{2} D
\end{cases}
\]

(13)

5. DETECTOR NOISE EFFECTS

The last source under consideration of residual differential piston phase is that generated by photon noise in our fringe tracking sensor. A recent work\(^{12}\) suggests the fact that current working interferometers are still detector-noise limited, or at least that the detector-noise contribution is not negligible at all. Yet in our approach we consider only photon-noise as far as our analysis is to be applied to Michelson interferometers with very large pupils. We have adopted the phase-tracking method described in Ref. 13 and the reader is referred to that work for details on it. According to Ref. 13 the expected phase variance under photon-noise limited (and hence the subscript \( pn \)) wideband fringe tracking is given by \( (s^2_\phi)_{pn} = \pi^2/(4V^2 N) \) where \( V \) is the visibility and \( N \) is the number of detected photo-electrons and implicitly contains time and wavelength bandwidth dependences. Assuming white photon-noise and that it passes through a boxcar averager with time constant \( \tau \) we can write:

\[
\Gamma_{pn}(f) = \tau (s^2_\phi)_{pn} = \frac{\pi}{\eta V^2 D^2} 10^{0.4m-10} \left( \frac{10^6 \text{nm}}{\Delta \lambda} \right)
\]

(14)

where \( \Gamma_{pn}(f) \) is the power spectrum associated with the photon noise in terms of the two pupil collecting surface \( 2D \) (in units of \([m]\)), the quantum efficiency of the fringe sensor \( \eta \), the magnitude \( m \) of the reference star and the bandwidth \( \Delta \lambda \) of the sensing light. The transfer function \( T_{pn} \) we must use with \( \Gamma_{pn} \) is, according to Ref. 6, the overall transfer function of the closed loop as it is assumed that the noise in the device enters through the fringe sensor and propagates across the integrator and feedback elements:

\[
|T_{pn}(f)|^2 = \left| \frac{H_e(f)H_i(f)}{1 + H_e(f)H_i(f)} \right|^2 \simeq \frac{1}{1 + (2\pi f \tau)^4/12}
\]

(15)

The residual variance due to the presence of photon noise \( (s^2_\phi)_{pn} \) within the servo system is then readily evaluated to be:

\[
(s^2_\phi)_{pn} = 2 \int_0^\infty df |T_{pn}(f)|^2 \Gamma_{pn}(f) = \frac{3^{1/4} \pi 10^{0.4m-10} \left( \frac{10^6 \text{pn}}{\Delta \lambda} \right)}{2 \eta V^2 D^2 \tau}
\]

(16)
6. APPLICATION: SKY COVERAGE FOR 8M-CLASS INTERFEROMETERS

We consider the configurations corresponding to Keck ($D = 10\text{m}$, $\Delta = 85\text{m}$), LBT ($D = 8.4\text{m}$, $\Delta = 14.4\text{m}$), VLTI UT1-UT2 ($D = 8.2\text{m}$, $\Delta = 57\text{m}$) and VLTI UT1-UT4 ($D = 8.2\text{m}$, $\Delta = 130\text{m}$). For Keck and VLTI we use available SCIDAR measurements for the atmospheric turbulence profiles above Mauna Kea\textsuperscript{14} and Cerro Paranal.\textsuperscript{15} In absence of atmospheric turbulence profiles for LBT we have considered a modified Hufnagel-Valley model.\textsuperscript{6} These atmospheric turbulence profiles have been normalized to the median $r_0(0.5\mu\text{m})$ values above each site: 20.4, 13.3 and 14.1 cm for the Keck,\textsuperscript{14} LBT\textsuperscript{16} and VLTI,\textsuperscript{17} respectively. Wind profiles in Ref. 15 and Ref. 18 are used for Keck and VLTI and due to the absence of such data for LBT, we assume the standard Bufton wind model with ground wind speed set to 5 m/s. Finally, to our knowledge there is no vertical $\theta$ profile so that we will assume a uniformly random wind direction profile along the atmosphere. In the same manner as we define the concepts of anisoplanatic angle and the high-order critical time constant we define the isopistonic angle $\theta_p$ and the pistonic critical time $\tau_p$. Thus, we define $\theta_p$ as the angular radius of a circular region where the anisopistonic error reduces the fringe visibility to no more than 80% of the unperturbed value. The 20% visibility reduction happens when the residual piston error $\sigma_p$ is about $\lambda/10$. The pistonic critical time $\tau_p$ is defined as the value at which $(\sigma_p^2) = 1$rad implying a $\sim40\%$ fringe visibility reduction. Following Ref. 6 the fractional sky coverage and the density number of stars of visual magnitude less or equal to $m_v$ are given by:

\[
\text{fractional sky coverage} = \pi \vartheta^2 \times \text{(average star density)}_{m_v} \\
\text{(average star density)}_{m_v} = 1.45 \exp(0.96 m_v) \quad \text{(stars/rad}^2) 
\]

where $\vartheta$ is the angular radius around the reference source within which the isopistonic error reduces the visibility by a tolerable value. The limiting magnitude $m_v$ will be determined by the trade-off between the maximum integration time before $(\sigma_p^2) = 10\text{rad}$ degrades $V$ and the minimum required time beyond which $(\sigma_p^2)_{\text{pn}}$ preserves $V$ to its desired value. Considering that in addition to the fringe visibility caused by differential piston there will be other sources of visibility reduction (e.g. differential tilt, high-orders correction effects, finite bandwidth, etc) we find appropriate to set our goal to achieve a visibility reduction of 50% due to differential piston. Taking $\vartheta = \theta_p$ leaves a budget of $0.94\text{ rad}^2$ to be distributed between $(\sigma_p^2) = 10\text{rad}$ and $(\sigma_p^2)_{\text{pn}}$ of which we arbitrarily allocate a 90% to the former and a 10% to the latter. This arbitrary allocation yields integration times for the fringe sensor ranging from $\sim6\text{ ms}$ to $\sim10\text{ ms}$. From here we proceed by assuming a fringe detector with a total throughput of 40% and sensing done in the V-band. The value for the limiting magnitude in the V-band together with the expected fractional sky coverage values are also provided in Table 1. Thus, while in the V-band the expected sky coverage is only a few per cent, in the K-band we should be able to cover the entire sky.

**Table 1.** Summary of the parameters and results of the estimates on sky coverage.

<table>
<thead>
<tr>
<th>Interferometer</th>
<th>K-BAND $\theta_p$ [arcsec]</th>
<th>$\tau_p$ [ms]</th>
<th>V-BAND $\theta_p$ [arcsec]</th>
<th>$\tau_p$ [ms]</th>
<th>Limiting $m_v$</th>
<th>K-band (%)</th>
<th>V-band (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Keck</td>
<td>24.3</td>
<td>29.2</td>
<td>6.00</td>
<td>7.30</td>
<td>18.4</td>
<td>100</td>
<td>18</td>
</tr>
<tr>
<td>LBT</td>
<td>21.9</td>
<td>44.3</td>
<td>5.43</td>
<td>11.1</td>
<td>18.5</td>
<td>100</td>
<td>19</td>
</tr>
<tr>
<td>VLTI</td>
<td>27.1</td>
<td>70.4</td>
<td>6.68</td>
<td>17.6</td>
<td>18.6</td>
<td>100</td>
<td>44</td>
</tr>
<tr>
<td>L$_0 = 50\text{ m}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>18.4</td>
<td>72</td>
<td>4.5</td>
</tr>
<tr>
<td>Keck</td>
<td>13.0</td>
<td>25.0</td>
<td>3.24</td>
<td>6.25</td>
<td>18.4</td>
<td>72</td>
<td>4.5</td>
</tr>
<tr>
<td>LBT</td>
<td>13.4</td>
<td>39.1</td>
<td>3.33</td>
<td>9.78</td>
<td>18.5</td>
<td>84</td>
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<tr>
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<td>11.3</td>
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<td>100</td>
<td>7.9</td>
</tr>
</tbody>
</table>
7. CONCLUSION

Our calculations on the effects of differential piston for adaptive optics interferometry show that the sky-coverage in the V-band is indeed quite small, while close to full coverage in the K-band. These results are very much dependent on the particular $C_2^s(h)$ and $v(h)$ profiles as well as on the outer scale. At this point we do not claim that a particular interferometer will be able to cover twice as much sky as the other since the profiles used for any of them have been obtained with very different resolutions thus affecting the exact sky-coverage values obtained. Instead, these values should be viewed as orders of magnitude so that we conclude that the new generation of optical ground-based interferometers should be able to cover the sky at a level of 20-40% in the V-band and at 100% in the K-band if only differential piston were present and assuming $L_0 = 20 m$, while for $L_0 = 50 m$ we still reach a 100% sky-coverage in the K-band while in the V-band we are limited to 5-10% sky coverage. Further considerations including the effects of residual tip-tilt correlation between the two pupils (the so-called differential tilt) as well as residual higher-order corrections are likely to further constrain the sky-coverage of these interferometers, a question which should be studied in more detail.

REFERENCES